Published by Institute of Physics Publishing for SISSA/ISAS
Received: *December 21, 2004*
Revised: *January 18, 2005*
Accepted: *February 7, 2005*

Quantum geometries of A_2

Girma Hailu

Jefferson Laboratory of Physics, Harvard University
Cambridge, MA 02138, U.S.A.
E-mail: hailu@physics.harvard.edu

Abstract: We solve $\mathcal{N} = 1$ supersymmetric A_2 type $\mathrm{U}(N)\times\mathrm{U}(N)$ matrix models obtained by deforming $\mathcal{N} = 2$ with symmetric tree level superpotentials of any degree exactly in the planar limit. These theories can be geometrically engineered from string theories by wrapping D-branes over Calabi-Yau threefolds and we construct the corresponding exact quantum geometries.

Keywords: Nonperturbative Effects, Gauge Symmetry, Supersymmetric Effective Theories, Matrix Models.

Contents

1. Introduction

A class of supersymmetric gauge theories with tree level superpotentials can be geometrically engineered from type IIA and type IIB string theories by wrapping D-branes over various cycles of Calabi-Yau threefolds. See [1]–[5] for instance. The quantum Calabi-Yau geometries can be studied using the related geometrically engineered gauge theories. Constructing the exact quantum geometries associated to product gauge group theories with general tree level superpotentials is a highly nontrivial problem. More recently, important connections between matrix models and supersymmetric gauge theories have been found by Dijkgraaf and Vafa. [6]–[8] It was also found by expansion [7] that the quantum Calabi-Yau geometry that engineers A_2 could be expressed in terms of two polynomials. In this note, we will give analytic proof and find these polynomials by solving the matrix models for $\mathcal{N}=1$ supersymmetric A_2 type $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge theories with symmetric tree level superpotentials of any degree in the planar limit and thus construct the corresponding exact quantum Calabi-Yau geometries.

First let us start with a general $\mathcal{N}=2$ supersymmetric $\prod_i \mathrm{U}(N_i)$ gauge theory with link chiral superfields Q_{ij} and $Q_{ji}=Q_{ij}^\dagger$ transforming as $(\square,\overline{\square})$ and $(\overline{\square},\square)$ respectively under $\mathrm{U}(N_i)\times\mathrm{U}(N_j)$ and the corresponding matrix model. Consider the tree level superpotential

$$W_{\mathrm{tree}}(\Phi,Q)=\sum_{i,j} s_{ij}\,\mathrm{Tr}\,Q_{ij}\Phi_j Q_{ji}+\sum_i \mathrm{Tr}\,W_i(\Phi_i)\,, \tag{1.1}$$

where Φ_i is the scalar chiral superfield associated with $\mathrm{U}(N_i)$ and the indices are ordered such that $s_{ij}=-s_{ji}=1$ with $j>i$ when the i^{th} and j^{th} gauge groups are linked and $s_{ij}=0$ otherwise. The first term in (1.1) comes from the superpotential of $\mathcal{N}=2$ with bifundamental hypermultiplets. The second term is a polynomial in each Φ_i and it will contain quadratic mass terms which break $\mathcal{N}=2$ down to $N=1$. This theory can be geometrically engineered from type IIB string theory with D3, D5 and D7 branes wrapped over various cycles of Calabi-Yau threefolds and also from type IIA string theory with D6 branes wrapped over Calabi-Yau threefolds. [1]–[5] The partition function is defined as

$$Z = \frac{1}{Z_0}\int \prod_i d\Phi_i \prod_{i<j} dQ_{ij}dQ_{ji}\, e^{-\frac{1}{g_s}W_{\rm tree}(\Phi,Q)} . \tag{1.2}$$

and normalized such that in terms of the eigenvalues $\lambda_{i,1},\ldots,\lambda_{i,I},\ldots\lambda_{iN_i}$ of Φ_i it becomes

$$Z = \int \prod_{i,I} d\lambda_{i,I}\exp(-S_{\rm eff}) , \tag{1.3}$$

with the effective action [9]

$$S_{\rm eff} = \frac{1}{g_s}\sum_{i,I} W_i(\lambda_{i,I}) - 2\sum_{i,I<J}\log|\lambda_{i,I}-\lambda_{i,J}| + \sum_{i<j,I,J}|s_{ij}|\log|\lambda_{i,I}-\lambda_{j,J}| . \tag{1.4}$$

Note that the small letter index i denotes the $i^{\rm th}$ gauge group and the upper letter index I denotes eigenvalues. The equations of motion are obtained by minimizing (1.4) with $\lambda_{i,I}$,

$$W_i'(\lambda_{i,I}) - 2g_s\sum_{J\neq I}\frac{1}{\lambda_{i,I}-\lambda_{i,J}} + g_s\sum_{j,J}|s_{ij}|\frac{1}{\lambda_{i,I}-\lambda_{j,J}} = 0 . \tag{1.5}$$

Let us introduce the resolvents,

$$w_i(x) = \frac{1}{N_i}\sum_{I=1}^{N_i}\frac{1}{\lambda_{i,I}-x} , \tag{1.6}$$

where x is complex. Note that $w_i(x)$ obey the asymptotic larger x behavior

$$w_i(x) \to -\frac{1}{x} . \tag{1.7}$$

The eignevalues are distributed on the real axis of x. We will consider the case in which the gauge symmetry in the low energy theory is unbroken in this note. This corresponds to the case in which each set of eignevalues $\lambda_{i,I}$ is separately distributed on a single interval $[a_i,\, b_i]$. The equations of motion (1.5) expressed in terms of the resolvents give

$$S_i(w_i(x+i0)+w_i(x-i0)) - \sum_j |s_{ij}|S_j w_j(x) + W_i'(x) = 0 \tag{1.8}$$

for $x\in[a_i,\, b_i]$ where

$$S_i \equiv g_s N_i . \tag{1.9}$$

Following Dijkgraaf and Vafa [6, 7, 8], S_i will be identified with the glueball superfields defined in terms of the gauge chiral superfields $W_{i\alpha}$ associated to the confining $\mathrm{SU}(N_i)$ subgroup of $\mathrm{U}(N_i)$ as

$$S_i = -\frac{1}{32\pi^2}\,\mathrm{Tr}\, W_i^\alpha W_{i\alpha} . \tag{1.10}$$

JHEP02(2005)017

In the large N limit, the eigenvalues are continuously distributed and each resolvent $w_i(x)$ can be written as a sum a regular function $w_{ir}(x)$ which is a particular solution of (1.8) and another function $w_{is}(x)$ which contains the singular part of $w_i(x)$,

$$w_i(x) = w_{ir}(x) + w_{is}(x)\,. \tag{1.11}$$

We can think of this as a substitution for $w_i(x)$ in terms of $w_{ir}(x) + w_{is}(x)$ where $w_{ir}(x)$ satisfies the regular equation (1.12) below and we will then solve for $w_{is}(x)$ such that the asymptotic behavior (1.7) is satisfied. We will find that $w_{is}(x)$ is the singular part of the resolvent.

Putting (1.11) in (1.8) and setting

$$2S_i w_{ir}(x) - \sum_j |s_{ij}| S_j w_{jr}(x) + W_i'(x) = 0\,, \tag{1.12}$$

we obtain

$$S_i(w_{is}(x+i0) + w_{is}(x-i0)) - \sum_j |s_{ij}| S_j w_{js}(x) = 0\,, \tag{1.13}$$

for x in the branch cut $[a_i,\, b_i]$ of $w_{is}(x)$. In the large N limit, the eigenvalues become continuous and we introduce the eigenvalue densities

$$\rho_i(\lambda) = \frac{1}{N_i} \sum_I \delta(\lambda - \lambda_{i,I})\,, \tag{1.14}$$

normalized such that $\int \rho_i(\lambda) d\lambda = 1$, and (1.6) becomes

$$w_i(x) = \int \frac{\rho_i(\lambda) d\lambda}{\lambda - x}\,. \tag{1.15}$$

Once $w_i(x)$ are found, (1.15) can be inverted to determine $\rho_i(\lambda)$ and

$$\rho_i(\lambda) = \frac{1}{2\pi i}(w_i(\lambda + i0) - w_i(\lambda - i0))\,. \tag{1.16}$$

The multi-matrix planar free energy can be conveniently written as

$$\mathcal{F}_0 = \frac{1}{2}\sum_i S_i \int d\lambda \rho_i(\lambda) W_i(\lambda) - \frac{1}{2}\sum_{i,j} C_{ij} S_i S_j \int d\lambda \rho_i(\lambda) \log|\lambda|\,, \tag{1.17}$$

where $C_{ij} = 2\delta_{ij} - |s_{ij}|$ is the Cartan matrix. We will not do free energy calculations in this note. The reason we have added this last paragraph is because we find the free energy given by (1.17) in terms of single integrals simpler and useful for doing calculations and we have not seen it in the literature on multi-matrix models. The derivation is given in appendix A.

JHEP02(2005)017

2. Quantum geometries of A_2

In this section we will explicitly construct the quantum Calabi-Yau geometries associated to $\mathcal{N}=1$ supersymmetric A_2 type $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge theories obtained by deforming $\mathcal{N}=2$ with symmetric tree level superpotentials of any degree and the gauge symmetry unbroken in the low energy theory. In the low energy theory, the $\mathrm{U}(1)$ subgroup of each $\mathrm{U}(N)$ decouples and the $\mathrm{SU}(N)$ subgroup confines. The most general asymptotically free product gauge theories of the type discussed in section 1 for the confining $\Pi_i\,\mathrm{SU}(N_i)$ subgroup with $\mathcal{N}=2$ supersymmetry and link chiral superfields in the bifundamental representation are constrained to be only of $A-D-E$ type Dynkin diagrams. See [10] for instance. The reason is that the condition of asymptotic freedom for the i^{th} gauge group can be written as $(2\delta_{ij}-\sum_{j\neq i}|s_{ij}|)N_j>0$ and this results in the constraint that all eigenvalues of the connectivity matrix $|s_{ij}|$ need to be less that 2 in order for the theory to be asymptotically free. Thus $(2\delta_{ij}-|s_{ij}|)$ is the Cartan matrix of $A-D-E$ type Dynkin diagrams and the most general asymptotically free such $\mathcal{N}=2$ product gauge theories with link chiral superfields in the bifundamental representation are of $A-D-E$ type. When the eignevalues of the connectivity matrix also contain 2, the beta function vanishes and theory is conformal and the diagram is that of affine $\hat{A}-\hat{D}-\hat{E}$ type.

Our interest is $\mathcal{N}=2$ supersymmetric A_2 type $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge theory deformed to $\mathcal{N}=1$ by symmetric tree level superpotentials with the gauge symmetry preserved in the low energy theory. This corresponds to two separate cuts for the resolvents associated to each gauge group in the matrix model. It follows from (1.16) that each branch cut is a square root branch cut. The regular parts of the resolvents $w_{1r}(x)$ and $w_{2r}(x)$ are solutions of the following equations which follow from (1.12) with i,j running over $1,2$,

$$2S_1w_{1r}(x)-S_2w_{2r}(x)+W_1'(x)=0\,,\qquad 2S_2w_{2r}(x)-S_1w_{1r}(x)+W_2'(x)=0\,. \tag{2.1}$$

The solutions are

$$w_{1r}(x)=-\frac{1}{3S_1}\left(2W_1'(x)+W_2'(x)\right),\qquad w_{2r}(x)=-\frac{1}{3S_2}\left(2W_2'(x)+W_1'(x)\right). \tag{2.2}$$

Now the tree level superpotential (1.1) becomes

$$W_{\text{tree}}(\Phi,Q)=\mathrm{Tr}\,Q_{12}\Phi_2Q_{21}-\mathrm{Tr}\,Q_{21}\Phi_1Q_{12}+\mathrm{Tr}\,W_1(\Phi_1)+\mathrm{Tr}\,W_2(\Phi_2)\,. \tag{2.3}$$

The classical equations of motion are

$$\begin{aligned} Q_{12}\Phi_2-\Phi_1Q_{12}=0\,,&\qquad Q_{21}\Phi_1-\Phi_2Q_{21}=0\,,\\ -Q_{12}Q_{21}+\frac{\partial W_1(\Phi_1)}{\partial\Phi_1}=0\,,&\quad Q_{21}Q_{12}+\frac{\partial W_2(\Phi_2)}{\partial\Phi_2}=0\,. \end{aligned} \tag{2.4}$$

Combining these equations, we can write

$$\begin{aligned} (X-S_1w_{1r})(X+S_1w_{1r}-S_2w_{2r})=0\,,&\qquad (X+S_2w_{2r})=0\,,\\ (Y-S_2w_{2r})(Y-S_1w_{1r}+S_2w_{2r})=0\,,&\qquad (Y+S_1w_{1r})=0\,, \end{aligned} \tag{2.5}$$

where $X = -Q_{21}Q_{12} - S_1 w_{1r} + S_2 w_{2r}$ and $Y = Q_{12}Q_{21} + S_1 w_{1r} - S_2 w_{2r}$. The singular classical spectral curve can be written in terms of a complex variable y as

$$(y + S_1 w_{1r}(x))(y - S_2 w_{2r}(x))(y - S_1 w_{1r}(x) + S_2 w_{2r}(x)) = 0\,. \tag{2.6}$$

The corresponding classical Calabi-Yau geometry is the singular threefold,

$$uv + (y + S_1 w_{1r}(x))(y - S_2 w_{2r}(x))(y - S_1 w_{1r}(x) + S_2 w_{2r}(x)) = 0\,, \tag{2.7}$$

which describes the A_2 fibration over the x plane, where u, v and y are complex coordinates. At the quantum level, the classical singularities are resolved and the spectral curve that describes the quantum resolution of the geometry is that of the resolved threefold and it should be given by (2.6) with the classical values of the resolvents replaced by the singular parts of the quantum resolvents,

$$\begin{aligned}(y + S_1 w_{1s}(x)))(y - S_2 w_{2s}(x))(y - S_1 w_{1s}(x) + S_2 w_{2s}(x)) =&\\ = (y - S_1 w_{1r}(x) + S_1 w_1(x)))(y + S_2 w_{2r}(x) - S_2 w_2(x))&\\ = (y + S_1 w_{1r}(x) - S_2 w_{2r}(x) - S_1 w_1(x) + S_2 w_2(x)) = 0\,.&\end{aligned} \tag{2.8}$$

Putting the decomposition given by (1.11) in (2.8) and using the classical solution given by (2.2) gives

$$(y + S_1 w_{1r}(x))(y - S_2 w_{2r}(x))(y - S_1 w_{1r}(x) + S_2 w_{2r}(x)) - f(x)\,y - g(x) = 0\,, \tag{2.9}$$

where

$$\begin{aligned}f(x) =& \; S_1^2 w_{1s}(x)^2 + S_2^2 w_{2s}(x)^2 - S_1 S_2 w_{1s}(x) w_{2s}(x) -\\ &- \frac{1}{3}(W_1'(x)^2 + W_2'(x)^2 + W_1'(x) W_2'(x))\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}g(x) =& \; S_1^2 S_2 w_{1s}(x)^2 w_{2s}(x) - S_1 S_2^2 w_{1s}(x) w_{2s}(x)^2 -\\ &- \frac{1}{27}(2W_1'(x)^3 - 2W_2'(x)^3 + 3W_1'(x)^2 W_2'(x) - 3W_1'(x) W_2'(x)^2)\,.\end{aligned} \tag{2.11}$$

The relation between Calabi-Yau geometries and matrix models was first found in [6]–[8] and the general classical and quantum curves as in the form (2.6) and (2.9) in terms of two general polynomial functions $f(x)$ and $g(x)$ was given in [7]. These polynomials describe the resolution of the singularities in the quantum theory and constructing them for general tree level superpotential is a highly nontrivial problem. Here we will construct the exact polynomials $f(x)$ and $g(x)$ that describe the quantum resolved geometry for $\mathrm{U}(N) \times \mathrm{U}(N)$ with symmetric tree level superpotentials $W_1(x)$ and $W_2(x) = W_1(-x)$ of any degree with the gauge symmetry preserved in the low energy theory.

The singular parts of the resolvents satisfy (1.13) which for the case of $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$ becomes

$$S_1(w_{1s}(x + i0) + w_{1s}(x - i0)) - S_2 w_{2s}(x) = 0 \qquad \text{for } x \in [a_1,\, b_1] \tag{2.12}$$

and

$$S_2(w_{2s}(x+i0)+w_{2s}(x-i0))-S_1w_{1s}(x)=0 \qquad \text{for } x\in[a_2,\,b_2]\,. \tag{2.13}$$

The resolvents satisfy two independent equations, one quadratic and the other cubic,

$$\begin{aligned}S_1^2w_1(x)^2+S_2^2w_2(x)^2-S_1S_2w_1(x)w_2(x)+S_1W_1'(x)w_1(x)+&\\ +S_2W_2'(x)w_2(x)+f_1(x)+f_2(x)\,=\,0&\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}S_1^2S_2w_1(x)^2w_2(x)-S_1S_2^2w_1(x)w_2(x)^2+S_1^2w_1(x)^2W_1'(x)+&\\ +S_1w_1(x)W_1'(x)^2+f_1(x)W_1'(x)-&\\ -g_1(x)-S_2^2w_2(x)^2W_2'(x)-S_2w_2(x)W_2'(x)^2-f_2(x)W_2'(x)+g_2(x)\,=\,0\,,&\end{aligned} \tag{2.15}$$

where

$$f_i(x)\equiv\frac{S_i}{N_i}\sum_I\frac{W_i'(x)-W_i'(\lambda_{iI})}{x-\lambda_{i,I}} \tag{2.16}$$

and

$$g_1(x)\equiv\frac{S_1S_2}{N_1N_2}\sum_{I,J}\frac{W_1'(x)-W_1'(\lambda_{1I})}{(\lambda_{1,I}-\lambda_{2,J})(x-\lambda_{1,I})}\,,\quad g_2(x)\equiv\frac{S_1S_2}{N_1N_2}\sum_{I,J}\frac{W_2'(x)-W_2'(\lambda_{2I})}{(\lambda_{2,I}-\lambda_{1,J})(x-\lambda_{2,I})}\,. \tag{2.17}$$

are polynomials. The quadratic equation for the $O(n)$ matrix model was obtained in [11] and the quadratic and cubic equations (2.14) and (2.15) for the A_2 model were obtained in [12]–[15]. We have also given a derivation in appendix B. The most general independent equations that $w_{1s}(x)$ and $w_{2s}(x)$ satisfy are at most cubic in either $w_{1s}(x)$ and $w_{2s}(x)$ or their combinations and there are three Riemann sheets with one cut for $w_{1s}(x)$ joining the first and the second sheets and a second cut for $w_{2s}(x)$ joining the second and third sheets.

In the large N limit, $w_i(x)$ can be expressed as in (1.11) which with (2.2) in (2.14) and (2.15) gives

$$S_1^2w_{1s}(x)^2+S_2^2w_{2s}(x)^2-S_1S_2w_{1s}(x)w_{2s}(x)=3p(x) \tag{2.18}$$

and

$$S_1^2S_2w_{1s}(x)^2w_{2s}(x)-S_1S_2^2w_{1s}(x)w_{2s}(x)^2=2q(x)\,, \tag{2.19}$$

where

$$p(x)=\frac{1}{9}\left(W_1'(x)^2+W_2'^2+W_1(x)'W_2'(x)-3f_1(x)-3f_2(x)\right) \tag{2.20}$$

and

$$\begin{aligned}q(x)\,=\,&\frac{4}{27}((W_1'(x)^3-W_2'(x)^3)+\frac{1}{18}(W_1'(x)^2W_2'(x)-W_1'(x)W_2'(x)^2)-\\ &-\frac{1}{2}(W_1'(x)f_1(x)-W_2'(x)f_2(x))-((W_1'(x)-W_2'(x))p(x)+\frac{1}{2}(g_1(x)-g_2(x))\,.\end{aligned} \tag{2.21}$$

Using (2.18)–(2.21) in (2.10) and (2.11) we have

$$f(x)=3p(x)-\frac{1}{3}(W_1'(x)^2+W_2'(x)^2+W_1'(x)W_2'(x)) \tag{2.22}$$

and

$$g(x) = 2q(x) - \frac{1}{27}(2W_1'(x)^3 - 2W_2'(x)^3 + 3W_1'(x)^2W_2'(x) - 3W_1'(x)W_2'(x)^2)\,. \qquad (2.23)$$

Note that $p(x)$ is a polynomial of degree $2n_1$ or $2n_2$ and $q(x)$ is a polynomial of degree $3n_1$ or $3n_2$ depending on whether $n_1 > n_2$ or not. Combining (2.18) and (2.19) gives

$$S_1^3 w_{1s}(x)^3 - 3p(x)S_1 w_{1s}(x) = -S_2^3 w_{2s}(x)^3 + 3p(x)S_2 w_{2s}(x) = 2q(x)\,. \qquad (2.24)$$

Our interest is the case in which the two gauge groups are the same and the potentials $W_1(x)$ and $W_2(x)$ have the same degree $n+1$ with $n_1 = n_2 = n$ and are symmetric about the origin such that $W_2(x) = W_1(-x)$. We will set $N_1 = N_2 = N$ from now on and the gauge symmetry is $\mathrm{U}(N) \times \mathrm{U}(N)$ and it will be preserved in the low energy theory. Thus we also have $S_1 = S_2 = g_s N \equiv S$. This corresponds to two separate cuts in $w_1(x)$ and $w_2(x)$ associated to each gauge group in the matrix model. We will set up the potentials $W_1(x)$ and $W_2(x)$ such that the branch cuts of $w_1(x)$ and $w_2(x)$ will be symmetrically on the positive and the negative real axis of x respectively and we have $\lambda_{1,I} > 0$ and $\lambda_{2,J} < 0$. The equations that $Sw_{1s}(x)$ and $-Sw_{2s}(x)$ satisfy are similar to that of the $O(n)$ matrix model investigated in [11] and we will use techniques developed in [11] to solve the $\mathrm{U}(N) \times \mathrm{U}(N)$ matrix model. We will impose appropriate boundary conditions that produce the desired properties described above. First we start with one of the solutions to the cubic equation (2.24) for $w_{1s}(x)$,

$$w_{1s}(x) = \frac{1}{S}\left(e^{-2\pi i/3}w_{s+}(x) + e^{2\pi i/3}w_{s-}(x)\right), \qquad (2.25)$$

where

$$w_{s\pm}(x) = \left(q(x) \mp \sqrt{q(x)^2 - p(x)^3}\right)^{1/3}. \qquad (2.26)$$

It follows from (2.26) that

$$p(x) = w_{s+}(x)w_{s-}(x)\,, \qquad (2.27)$$

$$q(x) = \frac{1}{2}\left(w_{s+}(x)^3 + w_{s-}(x)^3\right), \qquad (2.28)$$

and

$$\sqrt{q(x)^2 - p(x)^3} = \frac{1}{2}\left(w_{s+}(x)^3 - w_{s-}(x)^3\right). \qquad (2.29)$$

The second resolvent $w_{2s}(x)$ also follows as one of the three solutions to the cubic equation in (2.24) with appropriate boundary conditions to be imposed,

$$w_{2s}(x) = \frac{1}{S}\left(e^{-\pi i/3}w_{s+}(x) + e^{\pi i/3}w_{s-}(x)\right). \qquad (2.30)$$

The third solution to the cubic equation is a linear combination of the two resolvents.

The square root branch cuts in $w_{1s}(x)$ and $w_{2s}(x)$ come from $\sqrt{q(x)^2 - p(x)^3}$. In order to fulfill the constraint that the two branch cuts be symmetric, we need to have $a_2 = -b_1$ and $b_2 = -a_1$ so that $w_{1s}(x)$ will have its branch cut on $x \in [a\,, b]$ and $w_{2s}(x)$ on $x \in [-b\,, -a]$ with $b > a > 0$. This will be achieved by imposing the following symmetries which are extensions of the symmetries imposed in [11] for the $O(n)$ matrix model,

$$\begin{aligned} w_{s\pm}(x-i0) &= e^{\pm 2\pi i/3} w_{s\mp}(x+i0) \qquad \text{for } x \in [a,\, b]\,, & (2.31)\\ w_{s\pm}(x-i0) &= e^{\pm 4\pi i/3} w_{s\mp}(x+i0) \qquad \text{for } x \in [-b,\, -a] & (2.32)\\ w_{s+}(x) &= w_{s-}(-x)\,. & (2.33) \end{aligned}$$

It then follows from (2.25), (2.26), (2.30) and (2.33) that $w_{2s}(x) = -w_{1s}(-x)$. Note also that combining (2.33) with (2.29) implies that $q(0)^2 = p(0)^3$. The main reason for the choice of the symmetries (2.31)–(2.33) on $w_{1s}(x)$ and $w_{2s}(x)$ given in (2.25) and (2.30) is that we have for $x \in [a\,, b]$,

$$w_{1s}(x-i0) = \frac{1}{S}(w_{s+}(x+i0) + w_{s-}(x+i0))\,, \qquad w_{2s}(x-i0) = w_{2s}(x+i0)\,, \tag{2.34}$$

and for $x \in [-b\,, -a]$,

$$w_{2s}(x-i0) = -\frac{1}{S}(w_{s+}(x+i0) + w_{s-}(x+i0))\,, \qquad w_{1s}(x-i0) = w_{1s}(x+i0)\,. \tag{2.35}$$

Thus $w_{1s}(x)$ has a branch cut across $x \in [a\,, b]$ and no discontinuity across $x \in [-b\,, -a]$. On the other hand, $w_{2s}(x)$ has a branch cut across $x \in [-b\,, -a]$ and no branch cut across $x \in [a\,, b]$. This is exactly what we wanted.

It follows from (2.25), (2.30) and (2.33) that

$$w_{s\pm}(x) = -\frac{iS}{\sqrt{3}}\left(e^{-2\pi i/3} w_{1s}(\pm x) - e^{2\pi i/3} w_{1s}(\mp x)\right) = \frac{iS}{\sqrt{3}}\left(e^{-\pi i/3} w_{2s}(\pm x) - e^{\pi i/3} w_{2s}(\mp x)\right). \tag{2.36}$$

The asymptotic behavior of $w_i(x)$ given by (1.7) combined with (1.11) and (2.36) gives the asymptotic large x behaviors

$$w_{\pm}(x) \to \mp\frac{iS}{\sqrt{3}x}\,, \tag{2.37}$$

where

$$w_{\pm}(x) = w_{r\pm}(x) + w_{s\pm}(x) \tag{2.38}$$

and $w_{r\pm}(x)$ are given by the same expressions given in (2.36) for $w_{s\pm}(x)$ with $w_{is}(\pm x)$ replaced by $w_{ir}(\pm x)$. Noting that $Sw_{r\pm}(x)$ are independent of S, let us define

$$\Omega_{\pm}(x) \equiv \frac{\partial\,(Sw_{\pm})}{\partial S} = \frac{\partial\,(Sw_{s\pm})}{\partial S}\,. \tag{2.39}$$

It is then convenient to decompose $w_{s\pm}(x)$ as

$$w_{s\pm}(x) = \Omega_{\pm}(x) h_{\pm}(x)\,, \tag{2.40}$$

with $\Omega_{\pm}(x)$ obeying the same boundary conditions (2.31)–(2.33) as $w_{s\pm}(x)$ and having the same large x asymptotic behaviors given in (2.37) for $w_{\pm}(x)$. Note that because $\Omega_{\pm}(x)$ are obtained by taking first derivative of $Sw_{s\pm}(x)$ which have square root branch cuts, $\Omega_{\pm}(x)$ must have simple poles at the branch points. Moreover, because $\Omega_{\pm}(x)$ obey the boundary conditions given in (2.31)–(2.33) with the asymptotic behaviors given by (2.37), $\Omega_{+}(x)\Omega_{-}(x)$ is even in x and with the simple poles at $\pm a$ and $\pm b$, we can write it in its most general form as

$$\Omega_+(x)\Omega_-(x) = \frac{S^2}{3}\frac{x^2-e^2}{(x^2-a^2)(x^2-b^2)}\,, \tag{2.41}$$

where e is a constant and $\Omega_+(x)\Omega_-(x)$ has two zeros at $x = \pm e$. We will choose $x = +e$ to be a zero of $\Omega_+(x)$ and $x = -e$ to be a zero of $\Omega_-(x)$. Following [11], it is convenient to define functions that will simplify our notations,

$$g_\pm(x) = \frac{\sqrt{(x^2-a^2)(x^2-b^2)} \pm \frac{x}{e}\sqrt{(e^2-a^2)(e^2-b^2)}}{x^2-e^2}\,. \tag{2.42}$$

The functions $\Omega_\pm(x)$ that satisfy the above properties can then be written in their most general forms as

$$\Omega_\pm(x) = -\frac{i}{\sqrt{(x^2-a^2)(x^2-b^2)}}\left((x^2-e^2)(cg_\mp(x) \pm dx)\right)^{1/3}\,, \tag{2.43}$$

where c and d are constants. Putting (2.43) in (2.41), we obtain

$$\frac{a^2b^2}{e^2}c^2 - c^2x^2 - d^2e^2x^2 - 2\frac{cd}{e}\sqrt{(e^2-a^2)(e^2-b^2)}\,x^2 + d^2x^4 - \frac{1}{27}S^6(x^2-e^2)^2 = 0\,. \tag{2.44}$$

The constants d, c and e are expressed in terms of a and b using (2.44) at $x = \infty$, 0 and e which give

$$d = \frac{1}{3\sqrt{3}}S^3\,, \tag{2.45}$$

$$c = -\frac{2}{3\sqrt{3}}S^3\frac{\sqrt{(e^2-a^2)(e^2-b^2)}}{e - a^2b^2/e^3}\,, \tag{2.46}$$

and

$$e^4 + 2ab\sqrt{(e^2-a^2)(e^2-b^2)} - a^2b^2 = 0\,, \tag{2.47}$$

where the appropriate signs are chosen such that the desired asymptotic behaviors are produced.

It also follows from (2.40) and the constraint that $w_{s\pm}(x)$ and $\Omega_\pm(x)$ satisfy the same boundary conditions that

$$h_\pm(x - i0) = h_\mp(x + i0) \qquad \text{for } x \in [a,\, b] \text{ and } x \in [-b,\, -a]\,. \tag{2.48}$$

Thus $h_+(x) + h_-(x)$ is regular while $h_+(x) - h_-(x)$ has square root branch cuts across $x \in [a, b]$ and $x \in [-b, -a]$. Because $w_\pm(x)$ and $\Omega_\pm(x)$ have the asymptotic behaviors given in (2.37) and $w_{s\pm}(x)$ have the asymptotic behavior $\sim x^n$, $h_\pm(x)$ must have the large x behavior $\sim x^{n+1}$. We can write $h_\pm(x)$ that satisfies these constraints in the most general form as

$$h_\pm(x) = \sqrt{(x^2-a^2)(x^2-b^2)}\left(A(x^2)g_\pm(x) \pm xB(x^2)\right)\,, \tag{2.49}$$

where $A(x^2)$ and $B(x^2)$ are even polynomials of degree at most $n-2$ and $n-4$ respectively if n is even and each of degree at most $n-3$ if n is odd. We then obtain $w_{s\pm}(x)$ using (2.43) and (2.49) with (2.45)–(2.47) in (2.40),

$$w_{s\pm}(x) = -i\frac{1}{\sqrt{3}}S\left((x^2-e^2)\left(\frac{e^3}{ab}g_\mp(x) \pm x\right)\right)^{1/3}\left(A(x^2)g_\pm(x) \pm xB(x^2)\right)\,, \tag{2.50}$$

where $A(x^2)$, $B(x^2)$ and the constants a, b and e are calculated putting (2.2) and (2.50) in (2.38), making use of the asymptotic behaviors (2.37), and using the constraint given by (2.47) for any given tree level superpotentials $W_1(x)$ and $W_2(x)$ symmetric about the origin and the resolvents having separate cuts. With the resolvents completely determined in terms of the input parameters of the theory, we have solved the matrix model in the planar limit.

The polynomials $p(x)$ and $q(x)$ follow from (2.50) in (2.27) and (2.28), see appendix C for more details,

$$p(x) = \frac{S^2}{3}\left(\left(x^2 - \frac{a^2b^2}{e^2}\right)A(x^2)^2 + \left(\frac{e^3}{ab} - \frac{ab}{e}\right)x^2A(x^2)B(x^2) - (x^4 - e^2x^2)B(x^2)^2\right) \tag{2.51}$$

and

$$\begin{aligned} q(x) = \frac{i}{6\sqrt{3}}S^3\Bigg[&\left(\left(3\frac{a^2b^2}{e} - e^3\right)x^2 + 2\frac{a^3b^3}{e^3}\right)A(x^2)^3 + \\ &+ 3\left(2x^4 - \left(\frac{a^2b^2}{e^2} + e^2\right)x^2\right)A(x^2)^2B(x^2) + \\ &+ 3\left(\left(\frac{e^3}{ab} + \frac{a^2b^2}{e}\right)x^4 - 2abex^2\right)A(x^2)B(x^2)^2 + \\ &+ \left(2x^6 + \left(\frac{e^6}{a^2b^2} - 3e^2\right)x^4\right)B(x^2)^3\Bigg]. \end{aligned} \tag{2.52}$$

With $p(x)$ and $q(x)$ in hand, we have found the explicit forms of the quantum resolution functions putting (2.51), (2.52) and (2.2) in (2.22) and (2.23).

The final result for the spectral curve that describes quantum geometry follows from (2.2), (2.9)–(2.11), (2.51) and (2.52),

$$\begin{aligned} &\left(y - \frac{1}{3}(2W_1'(x) + W_2'(x)\right)\left(y + \frac{1}{3}(W_1'(x) + 2W_2'(x)\right) + \\ &\quad + \left(y + \frac{1}{3}(W_1'(x) - W_2'(x)\right) - f(x)\,y - g(x) = 0\,, \end{aligned} \tag{2.53}$$

where $f(x)$ and $g(x)$ are given by

$$\begin{aligned} f(x) &= S^2\left(\left(x^2 - \frac{a^2b^2}{e^2}\right)A(x^2)^2 + \left(\frac{e^3}{ab} - \frac{ab}{e}\right)x^2A(x^2)B(x^2) - (x^4 - e^2x^2)B(x^2)^2\right) - \\ &\quad - \frac{1}{3}(W_1'(x)^2 + W_2'(x)^2 + W_1'(x)W_2'(x))\,, \end{aligned} \tag{2.54}$$

$$\begin{aligned} g(x) &= \frac{i}{3\sqrt{3}}S^3\Bigg[\left(\left(3\frac{a^2b^2}{e} - e^3\right)x^2 + 2\frac{a^3b^3}{e^3}\right)A(x^2)^3 + \\ &\quad + 3\left(2x^4 - \left(\frac{a^2b^2}{e^2} + e^2\right)x^2\right)A(x^2)^2B(x^2) + \\ &\quad + 3\left(\left(\frac{e^3}{ab} + \frac{a^2b^2}{e}\right)x^4 - 2abex^2\right)A(x^2)B(x^2)^2 + \\ &\quad + \left(2x^6 + \left(\frac{e^6}{a^2b^2} - 3e^2\right)x^4\right)B(x^2)^3\Bigg] - \\ &\quad - \frac{1}{27}(2W_1'(x)^3 - 2W_2'(x)^3 + 3W_1'(x)^2W_2'(x) - 3W_1'(x)W_2'(x)^2)\,. \end{aligned} \tag{2.55}$$

The even polynomials $A(x^2)$ and $B(x^2)$ and the constants a, b and e in (2.54) and (2.55) are completely determined for any given symmetric tree level superpotentials such that the branch cuts of $w_1(x)$ and $w_2(x)$ are disconnected and on opposite sides of the origin making use of the relation given by (2.47) and the asymptotic behaviors of $w_\pm(x)$ given by (2.37).

Let us now apply our results to the simple case of symmetric quadratic potentials,

$$W_1(x) = \frac{1}{2}mx^2 - \alpha x \qquad \text{and} \qquad W_2(x) = \frac{1}{2}mx^2 + \alpha x\,, \tag{2.56}$$

where m and α are constants such that $w_1(x)$ and $w_2(x)$ have non overlapping branch cuts so that the gauge symmetry is unbroken in the low energy theory. The regular parts of the resolvents $w_{1r}(x)$ and $w_{2r}(x)$ follow from (2.56) in (2.2),

$$w_{1r}(x) = -\frac{1}{3S}\left(3mx - \alpha\right), \qquad w_{2r}(x) = -\frac{1}{3S}\left(3mx + \alpha\right). \tag{2.57}$$

Now (2.57) in (2.36) with w_{is} and $w_{s\pm}$ replaced by w_{ir} and $w_{r\pm}$ give

$$w_{r\pm}(x) = \mp\frac{i}{\sqrt{3}}mx - \frac{1}{3}\alpha\,. \tag{2.58}$$

In this case, the asymptotic behaviors of $w_\pm(x)$ require that $B(x^2) = 0$ and $A(x^2) = A$, where A is a constant, which with (2.50) and (2.58) in (2.38) give

$$w_\pm(x) = \mp\frac{i}{\sqrt{3}}mx - \frac{1}{3}\alpha - i\frac{1}{\sqrt{3}}AS\left(\left(x^2 - e^2\right)\left(\frac{e^3}{ab}g_\mp(x) \pm x\right)\right)^{1/3} g_\pm(x)\,. \tag{2.59}$$

Note that there are a total of four unknown parameters A, a, b and e. Demanding that $w_\pm(x)$ in (2.59) obey the asymptotic large x limits given by (2.37) gives three equations and we have one additional constraint among a, b and e given by (2.47). The asymptotic limits (2.37) on (2.59) give the following relations

$$A = -\frac{m}{S}\,, \tag{2.60}$$

$$e = -i\sqrt{3}\frac{m}{\alpha}ab\,, \tag{2.61}$$

and

$$m(a^2 + b^2) + 4\frac{m^3}{\alpha^2}a^2b^2 - 6\frac{m^7}{\alpha^6}a^4b^4 = 2S\,. \tag{2.62}$$

Combining (2.47) and (2.62), we obtain a simple expression for the sum of the squares of the locations of the branch points,

$$a^2 + b^2 = 18\frac{S}{m} + 2\frac{\alpha^2}{m^2}\,. \tag{2.63}$$

Explicit expressions for the locations of the branch points $\pm a$ and $\pm b$ and the constant parameter e are given in appendix D. The functions $f(x)$ and $g(x)$ that parameterize the quantum resolution of the classical Calabi-Yau singularities are also given in appendix E. The constants a, b and e are all completely determined in terms of the parameters of the theory m, α and S through (D.3), (D.4) and (D.5). Note also that a, b and e have nice relations. The constants a and b are real for real S, m and α as we demanded and the magnitudes of a and b become larger as the critical points of the potentials $x = \pm\alpha/m$ get further away from the origin. Moreover, e is pure imaginary and nonzero for $a \neq 0$. Our solution describes a theory in which the gauge symmetry in the low energy theory is preserved and α/m is such that the two branch cuts, one from $w_1(x)$ and the other from $w_2(x)$ are disconnected with $b > a > 0$. As the parameters of the theory α, m and S are varied, the locations of the branch points move on the real axis of x and this is related to a movement of e on the imaginary axis of x.

3. Conclusion

In conclusion, matrix models in combination with supersymmetric gauge theories provide very powerful tools that allow us to study exact nonperturbative physics for systems involving quite general tree level superpotentials where symmetries and holomorphy alone are not enough. On the other hand, a class of supersymmetric gauge theories with tree level superpotentials can be geometrically engineered in type IIA and type IIB string theories by wrapping D-branes over various cycles of Calabi-Yau threefolds. The singularities in the classical Calabi-Yau geometry are resolved by quantum effects. In this note we have used the combined power of supersymmetry and matrix models to construct the exact quantum Calabi-Yau geometries associated to $\mathcal{N} = 1$ supersymmetric A_2 type $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge theories with quite general symmetric tree level polynomial superpotentials of any degree. Even though our interest in this note was the construction of the quantum geometries, our exact results could be used to compute the free energies in the planar limit and the exact nonperturbative dynamical superpotentials of A_2.

Acknowledgments

Part of this research was supported by the Department of Energy under grant number DE-FG02-91ER40654.

A. Multi-matrix free energy integral

Here we give the derivation of (1.17) which gives the multi-matrix planar free energy in terms of simple single integrals. First the free energy in the large N limit can be read off from the effective action given by (1.4) and it is

$$\begin{aligned}\mathcal{F}_0 = &\sum_i S_i \int d\lambda \rho_i(\lambda) W_i(\lambda) - \sum_i S_i^2 \int\int d\lambda d\mu \rho_i(\lambda)\rho_i(\mu) \log|\lambda - \mu| + \\ &+ \sum_{i<j} |s_{ij}| S_i S_j \int\int d\lambda d\mu \rho_i(\lambda)\rho_j(\mu) \log|\lambda - \mu| \,. \qquad \text{(A.1)}\end{aligned}$$

Our notation is such that in the planar limit the free energy is related to the partition function via $Z = e^{-\mathcal{F}_0/g_s^2}$. See [16] for more about our notation. The first two terms come from each cut separately while the last term is due to interactions between different sets.

Taking the large N limit of (1.5) and integrating the equation of motion over λ gives

$$\begin{aligned} S_i \int d\mu \rho_i(\mu) \log|\lambda - \mu| = & \frac{1}{2} W_i(\lambda) + S_i \int d\mu \rho_i(\mu) \log|\mu| + \\ & + \frac{1}{2} \sum_j |s_{ij}| S_j \int d\mu \rho_j(\mu) \log|\lambda - \mu| - \\ & - \frac{1}{2} \sum_j |s_{ij}| S_j \int d\mu \rho_j(\mu) \log|\mu| \,. \end{aligned} \tag{A.2}$$

Substituting (A.2) for $S_i \int d\mu \rho_i(\mu) \log|\lambda - \mu|$ in the second term of (A.1), remembering that the eigenvalue densities are normalized such that $\int \rho_i(\lambda) d\lambda = 1$, and simplifying we obtain

$$\mathcal{F}_0 = \frac{1}{2} \sum_i S_i \int d\lambda \rho_i(\lambda) W_i(\lambda) - \frac{1}{2} \sum_{i,j} C_{ij} S_i S_j \int d\lambda \rho_i(\lambda) \log|\lambda| \,, \tag{A.3}$$

where $C_{ij} = 2\delta_{ij} - |s_{ij}|$ is the Cartan matrix.

B. The quadratic and the cubic equations

Here we give a derivation of the quadratic and the cubic equations. The equations of motion of the eigenvalues for the $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$ matrix model are

$$\frac{1}{g_s} W_1'(\lambda_{1,I}) - 2 \sum_{J \neq I} \frac{1}{\lambda_{1,I} - \lambda_{1,J}} + \sum_J \frac{1}{\lambda_{1,I} - \lambda_{2,J}} = 0 \,, \tag{B.1}$$

$$\frac{1}{g_s} W_2'(\lambda_{2,I}) - 2 \sum_{J \neq I} \frac{1}{\lambda_{2,I} - \lambda_{2,J}} + \sum_J \frac{1}{\lambda_{2,I} - \lambda_{1,J}} = 0 \,. \tag{B.2}$$

Squaring the resolvent $w_1(x)$ defined by (1.6),

$$w_1(x)^2 = \frac{1}{N_1} w_1'(x) + \frac{2}{N_1^2} \sum_I \frac{1}{\lambda_{1,I} - x} \sum_{J \neq I} \frac{1}{\lambda_{1,I} - \lambda_{1,J}} \,. \tag{B.3}$$

Using the first equation of motion (B.1) to substitute for $\sum_{J \neq I} \frac{1}{\lambda_{1,I} - \lambda_{1,J}}$ in (B.3),

$$w_1(x)^2 = \frac{1}{N_1} w_1'(x) - \frac{1}{N_1 S_1} \sum_I \frac{W_1'(\lambda_{1I})}{\lambda_{1,I} - x} - \frac{1}{N_1^2} \sum_{I,J} \frac{1}{\lambda_{1,I} - x} \frac{1}{\lambda_{1,I} - \lambda_{2,J}} \,. \tag{B.4}$$

But

$$\begin{aligned} & \frac{1}{N_1^2} \sum_{I,J} \frac{1}{\lambda_{1,I} - x} \frac{1}{\lambda_{1,I} - \lambda_{2,J}} = \\ & \quad = -\frac{N_2}{N1} w_1(x) w_2(x) + \frac{1}{N_1^2} \sum_J \frac{1}{\lambda_{2,J} - x} \sum_I \frac{1}{\lambda_{1,I} - \lambda_{2,J}} . \end{aligned} \tag{B.5}$$

Using (B.2) to substitute for $\sum_{I \neq J} \frac{1}{\lambda_{1,I} - \lambda_{2,J}}$ in (B.5),

$$\begin{aligned} \frac{1}{N_1^2} \sum_{I,J} \frac{1}{\lambda_{1,I} - x} \frac{1}{\lambda_{1,I} - \lambda_{2,J}} = & -\frac{N_2}{N1} w_1(x) w_2(x) + \frac{1}{N_1 S_1} \sum_J \frac{W_2'(\lambda_{2J})}{\lambda_{2,J} - x} - \\ & - \frac{2}{N_1^2} \sum_J \frac{1}{\lambda_{2,J} - x} \sum_{I \neq J} \frac{1}{\lambda_{2,I} - \lambda_{2,I}} \,. \end{aligned} \tag{B.6}$$

Next squaring $w_2(x)$ we write the last term in (B.5) as

$$\frac{2}{N_1^2}\sum_J \frac{1}{\lambda_{2,J}-x}\sum_{I\neq J}\frac{1}{\lambda_{2,I}-\lambda_{2,I}} = \frac{N_2^2}{N_1^2}w_2(x)^2 - \frac{N_2}{N_1^2}w_2'(x)\,. \tag{B.7}$$

Using (B.6) and (B.7) in (B.4),

$$\begin{aligned} w_1(x)^2 - \frac{1}{N_1}w_1'(x) + \frac{1}{N_1 S_1}\sum_I \frac{W_1'(\lambda_{1I})}{\lambda_{1,I}-x} - \frac{N_2}{N1}w_1(x)w_2(x)+ \\ +\frac{1}{N_1 S_1}\sum_J \frac{W_2'(\lambda_{2J})}{\lambda_{2,J}-x} + \frac{N_2^2}{N_1^2}w_2(x)^2 - \frac{N_2}{N_1^2}w_2'(x) \,=\, 0\,. \end{aligned} \tag{B.8}$$

But we can write

$$\frac{1}{N_1 S_1}\sum_I \frac{W_1'(\lambda_{1I})}{\lambda_{1,I}-x} = \frac{1}{S_1}W_1'(x)w_1(x) + \frac{1}{S_1^2}f_1(x) \tag{B.9}$$

and

$$\frac{1}{N_1 S_1}\sum_I \frac{W_2'(\lambda_{2I})}{\lambda_{2,I}-x} = \frac{N_2}{N_1 S_1}W_2'(x)w_2(x) + \frac{N_2}{S_1 S_2 N_1}f_2(x)\,, \tag{B.10}$$

where

$$f_i(x) \equiv \frac{S_i}{N_i}\sum_I \frac{W_i'(x)-W_i'(\lambda_{iI})}{x-\lambda_{i,I}}\,. \tag{B.11}$$

Putting (B.9) and (B.10) in (B.8) and discarding the $\frac{1}{N_i}w_i'(x)$ terms in the large N limit gives the quadratic equation

$$\begin{aligned} S_1^2 w_1(x)^2 + S_2^2 w_2(x)^2 - S_1 S_2 w_1(x)w_2(x) + S_1 W_1'(x)w_1(x) + \\ + S_2 W_2'(x)w_2(x) + f_1(x) + f_2(x) \,=\, 0\,. \end{aligned} \tag{B.12}$$

Repeating a similar procedure as above starting with $w_1(x)^2 w_2(x)$ one obtains the cubic equation

$$\begin{aligned} S_1^2 S_2 w_1(x)^2 w_2(x) - S_1 S_2^2 w_1(x) w_2(x)^2 + \\ + S_1^2 w_1(x)^2 W_1'(x) + S_1 w_1(x) W_1'(x)^2 + f_1(x)W_1'(x) - \\ - g_1(x) - S_2^2 w_2(x)^2 W_2'(x) - S_2 w_2(x) W_2'(x)^2 - f_2(x)W_2'(x) + g_2(x) \,=\, 0\,, \end{aligned} \tag{B.13}$$

where

$$g_1(x) \equiv \frac{S_1 S_2}{N_1 N_2}\sum_{I,J}\frac{W_1'(x)-W_1'(\lambda_{1I})}{(\lambda_{1,I}-\lambda_{2,J})(x-\lambda_{1,I})}\,,\quad g_2(x) \equiv \frac{S_1 S_2}{N_1 N_2}\sum_{I,J}\frac{W_2'(x)-W_2'(\lambda_{2I})}{(\lambda_{2,I}-\lambda_{1,J})(x-\lambda_{2,I})}\,. \tag{B.14}$$

JHEP02(2005)017

C. The polynomials $p(x)$ and $q(x)$

Here we prove that the $p(x)$ and $q(x)$ that follow from the singular parts of the quantum resolvents given by (2.50) are indeed polynomials and their expressions are given. First the polynomial $p(x)$ is easily constructed. Using the decomposition given in (2.40) with (2.41) and (2.49) in (2.27) gives

$$p(x) = \frac{S^2}{3}(x^2-e^2)\left(g_+(x)g_-(x)A(x^2)^2 - x(g_+(x)-g_-(x))A(x^2)B(x^2) - x^2B(x^2)^2\right). \tag{C.1}$$

But (2.42) gives

$$(x^2-e^2)(g_+(x)g_-(x)) = x^2 - \frac{a^2b^2}{e^2} \text{ and } (x^2-e^2)(g_+(x)-g_-(x)) = 2\frac{x}{e}\sqrt{(e^2-a^2)(e^2-b^2)}\,. \tag{C.2}$$

Moreover, (2.47) gives

$$\sqrt{(e^2-a^2)(e^2-b^2)} = \frac{1}{2}\left(ab - \frac{e^4}{ab}\right). \tag{C.3}$$

Putting (C.2) and (C.3) in (C.1),

$$p(x) = \frac{S^2}{3}\left(\left(x^2 - \frac{a^2b^2}{e^2}\right)A(x^2)^2 + \left(\frac{e^3}{ab} - \frac{ab}{e}\right)x^2A(x^2)B(x^2) - (x^4 - e^2x^2)B(x^2)^2\right). \tag{C.4}$$

For $q(x)$ we start with (2.50) in (2.28) which gives

$$\begin{aligned} q(x) = {}& \frac{i}{6\sqrt{3}}S^3(x^2-e^2)\times \\ & \times\Bigg[\left(\frac{e^3}{ab}g_+(x)g_-(x)(g_+(x)^2+g_-(x)^2) + x(g_+(x)^3 - g_-(x)^3)\right)A(x^2)^3 + \\ & + 3x\Big(\frac{e^3}{ab}g_+(x)g_-(x)(g_+(x)-g_-(x)) + x(g_+(x)^2+g_-(x)^2)\Big)A(x^2)^2B(x^2) + \\ & + 3x^2\Big(2\frac{e^3}{ab}g_+(x)g_-(x) + x(g_+(x)-g_-(x))\Big)A(x^2)B(x^2)^2 - \\ & - x^3\Big(\frac{e^3}{ab}(g_+(x)-g_-(x)) - 2x\Big)B(x^2)^3\Bigg]. \end{aligned} \tag{C.5}$$

After some algebra involving $g_\pm(x)$ and also making use of the constraint given by (2.47), we obtain

$$\begin{aligned} q(x) = {}& \frac{i}{6\sqrt{3}}S^3\Bigg[\left(\left(3\frac{a^2b^2}{e} - e^3\right)x^2 + 2\frac{a^3b^3}{e^3}\right)A(x^2)^3 + \\ & + 3\left(2x^4 - \left(\frac{a^2b^2}{e^2} + e^2\right)x^2\right)A(x^2)^2B(x^2) + \\ & + 3\left(\left(\frac{e^3}{ab} + \frac{a^2b^2}{e}\right)x^4 - 2abex^2\right)A(x^2)B(x^2)^2 + \\ & + \left(2x^6 + \left(\frac{e^6}{a^2b^2} - 3e^2\right)x^4\right)B(x^2)^3\Bigg]. \end{aligned} \tag{C.6}$$

Thus the quantum resolvents indeed produce polynomials $p(x)$ and $q(x)$ in terms of two other even polynomials $A(x^2)$ and $B(x^2)$ which are specific to the tree level superpotentials $W_1(x)$ and $W_2(x)$.

D. Locations of branch points

Here we give explicit expressions for the locations of the branch points and the imaginary constant parameter e for the example of symmetric quadratic tree level superpotentials we discussed at the end of section 2. First (2.47) gives

$$e^8 - 6a^2b^2e^4 + 4a^2b^2(a^2+b^2)e^2 - 3a^4b^4 = 0\,. \tag{D.1}$$

Putting (2.61) in (D.1),

$$27\frac{m^8}{\alpha^8}a^4b^4 - 18\frac{m^4}{\alpha^4}a^2b^2 - 4\frac{m^2}{\alpha^2}(a^2+b^2) - 1 = 0\,. \tag{D.2}$$

Finally, solving (2.63) and (D.2) simultaneously and remembering that the phases and magnitudes of $\pm a$ and $\pm b$ are such that $b > a > 0$, we obtain

$$a = \sqrt{9\frac{S}{m} + \frac{\alpha^2}{m^2} - \sqrt{\frac{2}{3}\frac{\alpha^4}{m^4} + 18\frac{\alpha^2 S}{m^3} + 81\frac{S^2}{m^2} - \sqrt{\frac{4}{9}\frac{\alpha^8}{m^8} + \frac{8}{3}\frac{\alpha^6 S}{m^7}}}} \tag{D.3}$$

and

$$b = \sqrt{9\frac{S}{m} + \frac{\alpha^2}{m^2} + \sqrt{\frac{2}{3}\frac{\alpha^4}{m^4} + 18\frac{\alpha^2 S}{m^3} + 81\frac{S^2}{m^2} - \sqrt{\frac{4}{9}\frac{\alpha^8}{m^8} + \frac{8}{3}\frac{\alpha^6 S}{m^7}}}}\,. \tag{D.4}$$

Putting (D.3) and (D.4) in (2.61), we also obtain e,

$$e = -i\sqrt{\frac{\alpha^2}{m^2} + \sqrt{4\frac{\alpha^4}{m^4} + 24\frac{\alpha^2 S}{m^3}}}\,. \tag{D.5}$$

Note that a, b and e are all completely determined in terms of the parameters of the theory m, α and S through (D.3), (D.4) and (D.5). The locations of the branch points $\pm a$ and $\pm b$, the branch cuts, and the pure imaginary constant e are schematically shown in figure 1. As the parameters α/m and S/m are increased, the branch cuts and the imaginary parameter e get further away from the origin.

E. Quantum resolution functions

Here we write down the functions that describe the quantum resolution of the classical singularities of the Calabi-Yau geometry for the simple example of symmetric quadratic tree level superpotentials we applied our results at the end of section 2. First $p(x)$ and $q(x)$ are obtained using A, e, a and b found in (2.60), (2.61), (D.3) and (D.4) in (2.51) and (2.52),

$$p(x) = \frac{1}{3}m^2x^2 + \frac{1}{9}\alpha^2\,, \tag{E.1}$$

$$q(x) = -\frac{1}{3\sqrt{3}}\sqrt{\alpha^3(\alpha^2+6mS)(\alpha+2\sqrt{\alpha^2+6mS})}\,x^2 - \frac{1}{27}\alpha^3\,. \tag{E.2}$$

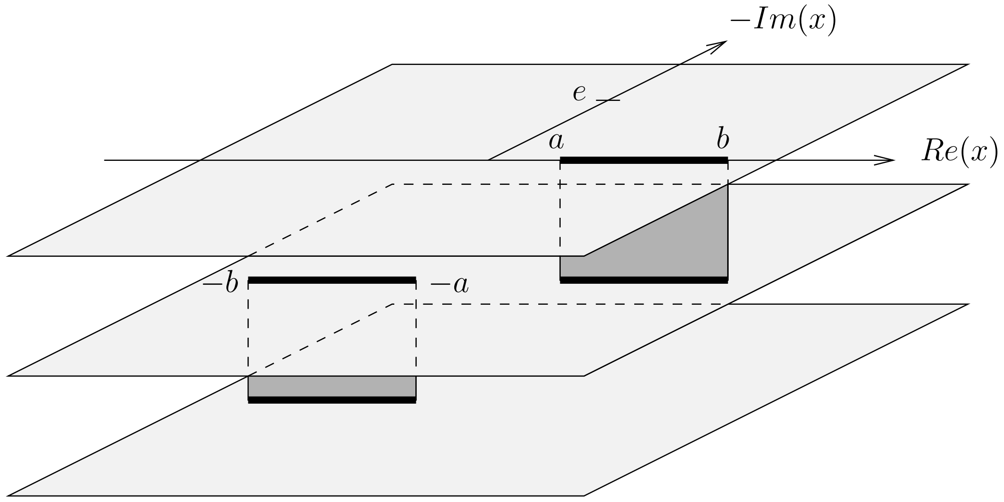

Figure 1: The three sheets, the two branch cuts, and e. The branch points are $\pm a$ and $\pm b$. The cuts are on the real axis of x and the one across $[a,\, b]$ is for $w_1(x)$ and it joins the first and the second sheets, the cut across $[-b,\, -a]$ is for $w_2(x)$ and it joins the second and the third sheets. The constant e is pure imaginary and its relation to a and b is given by (2.61).

Putting (2.56), (E.1) and (E.2) in (2.10) and (2.11), we obtain

$$f(x) = 0\,, \tag{E.3}$$

$$g(x) = \left(\frac{2}{3}\alpha m^2 - \frac{2}{3\sqrt{3}}\sqrt{\alpha^3(\alpha^2+6mS)(\alpha+2\sqrt{\alpha^2+6mS})}\right)x^2 - \frac{4}{27}\alpha^3\,. \tag{E.4}$$

References

[1] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

[2] C. Vafa, *Superstrings and topological strings at large-N*, *J. Math. Phys.* **42** (2001) 2798 [hep-th/0008142].

[3] S. Kachru, S. Katz, A.E. Lawrence and J. McGreevy, *Open string instantons and superpotentials*, *Phys. Rev.* **D 62** (2000) 026001 [hep-th/9912151].

[4] F. Cachazo, K.A. Intriligator and C. Vafa, *A large-N duality via a geometric transition*, *Nucl. Phys.* **B 603** (2001) 3 [hep-th/0103067].

[5] F. Cachazo, B. Fiol, K.A. Intriligator, S. Katz and C. Vafa, *A geometric unification of dualities*, *Nucl. Phys.* **B 628** (2002) 3 [hep-th/0110028].

[6] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[7] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106].

[8] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

JHEP02(2005)017

[9] I.K. Kostov, *Gauge invariant matrix model for the A-D-E closed strings*, *Phys. Lett.* **B 297** (1992) 74 [hep-th/9208053].

[10] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of 4d $N = 2$ gauge theories, I*, *Adv. Theor. Math. Phys.* **1** (1998) 53 [hep-th/9706110].

[11] B. Eynard and J. Zinn-Justin, *The $O(n)$ model on a random surface: critical points and large order behavior*, *Nucl. Phys.* **B 386** (1992) 558 [hep-th/9204082];
B. Eynard and C. Kristjansen, *Exact solution of the $O(n)$ model on a random lattice*, *Nucl. Phys.* **B 455** (1995) 577 [hep-th/9506193].

[12] C.I. Lazaroiu, *Holomorphic matrix models*, *J. High Energy Phys.* **05** (2003) 044 [hep-th/0303008].

[13] A. Klemm, K. Landsteiner, C.I. Lazaroiu and I. Runkel, *Constructing gauge theory geometries from matrix models*, *J. High Energy Phys.* **05** (2003) 066 [hep-th/0303032].

[14] S.G. Naculich, H.J. Schnitzer and N. Wyllard, *Cubic curves from matrix models and generalized Konishi anomalies*, *J. High Energy Phys.* **08** (2003) 021 [hep-th/0303268]; *Matrix-model description of $N = 2$ gauge theories with non- hyperelliptic Seiberg-Witten curves*, *Nucl. Phys.* **B 674** (2003) 37 [hep-th/0305263].

[15] R. Casero and E. Trincherini, *Quivers via anomaly chains*, *J. High Energy Phys.* **09** (2003) 041 [hep-th/0304123]; *Phases and geometry of the $N = 1$ A_2 quiver gauge theory and matrix models*, *J. High Energy Phys.* **09** (2003) 063 [hep-th/0307054].

[16] G. Hailu and H. Georgi, *On exact superpotentials, free energies and matrix models*, *J. High Energy Phys.* **02** (2004) 038 [hep-th/0401101].

JHEP02(2005)017